2019-Sep-12 (pre-lecture)

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MATB44H3 - 2019

Outline:

- More substitutions
- Exact differentials
- Integrating Factors

Last time:

We defined a separable first order ODE

$$\dot{x}(t) = g(t)f(x) \quad \left(\text{or equivalently } g(t)dt + f(x)dx = 0.\right)$$

We showed how to solve these by integrating $\int g(t)dt + \int f(x)dx = 0$.

We also defined homogeneous functions $f(tx, ty) = t^n f(x,y)$.

And showed that an ODE $P(x,y)dx + Q(x,y)dy = 0$ with homogeneous coefficients $P(x,y)$ and $Q(x,y)$ of the same order can be solved by making a substitution $y = ux$, $dy = u\,dx + x\,du$, which turns it into a separable ODE.

Linear Coefficients

We label ODEs of the form $(a_1x + b_1y + c_1)dx + (a_2x + b_2y + c_2)dy = 0$ as having linear coefficients.

We call them linear because $a_1x + b_1y + c_1 = 0$

$a_2x + b_2y + c_2 = 0$

define lines in the XY-plane.

What do we know how to solve? 1. separable equations

2. homogeneous coefficients

transform

We want to transform linear coefficients into

Let's start with an easy example. Suppose $c_1 = c_2 = 0$.

Then $(a_1x + b_1y)$ and $(a_2x + b_2y)$ are already homogeneous.

Let $y = ux$, $dy = u\,dx + x\,du$

$\Rightarrow (a_1 x + b_1 ux)dx + (a_2 x + b_2 ux)(u\,d\cancel{x} + x\,du) = 0$

$\Rightarrow [a_1 x + b_1 ux + a_2 ux + b_2 u^2 x]dx + [a_2 x^2 + b_2 ux^2]du = 0$

$\Rightarrow x(a_1 + b_1 u + a_2 u + b_2 u^2)dx + x^2(a_2 + b_2 u)du = 0$

If $x \neq 0$, and $a_1 + b_1 u + a_2 u + b_2 u^2 \neq 0$,

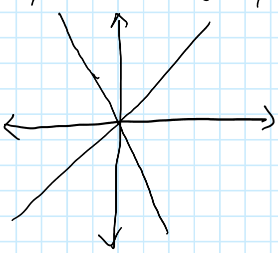
$$\frac{1}{x}dx + \frac{a_2 + b_2 u}{a_1 + b_1 u + a_2 u + b_2 u^2}du = 0. \quad \leftarrow \text{separable}$$

What was the key? Linear coefficients without constant terms are homogeneous.

Let's look at the $\mathbb{R}^2$ geometry of an example.

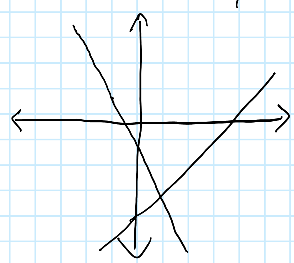
$(2x+y)dx + (x-y)dy = 0$

$$\left.\begin{matrix} 2x + y = 0 \\ x - y = 0 \end{matrix}\right\} \Rightarrow \left\{\begin{matrix} y = -2x \\ y = x \end{matrix}\right.$$



$(2x+y+1)dx + (x-y-4)dy = 0$

$$\left.\begin{matrix} 2x + y + 1 = 0 \\ x - y - 4 = 0 \end{matrix}\right\} \Rightarrow \left\{\begin{matrix} y = -2x - 1 \\ y = x - 4 \end{matrix}\right.$$



Having no constant terms means the lines intersect at the origin.

Can we change the coordinate system so the lines intersect at the origin?

Method 1: Find the intersection point and translate it.

$$\left.\begin{matrix} y = -2x - 1 \\ y = x - 4 \end{matrix}\right\} \Rightarrow \begin{matrix} -2x - 1 = x - 4 \\ 3x = 3 \\ x = 1 \\ y = -3 \end{matrix}$$

Let $x = \bar{x} + 1$ $\quad dx = d\bar{x}$

$y = \bar{y} - 3$ $\quad dy = d\bar{y}$

$(2x + y + 1)dx + (x - y - 4)dy = 0$

$(2\bar{x} + 2 + \bar{y} - 3 + 1)d\bar{x} + (\bar{x} + 1 - \bar{y} + 3 - 4)d\bar{y} = 0$

homogeneous $\longrightarrow (2\bar{x} + \bar{y})d\bar{x} + (\bar{x} - \bar{y})d\bar{y} = 0$

Method 2: Use the two lines as the new coordinate system

Let $u = 2x + y + 1$ $\quad du = 2dx + dy$

$v = x - y - 4$ $\quad dv = dx - dy$

$du + dv = 3dx$

$dx = \frac{du + dv}{3}$ $\quad dy = \frac{du - 2dv}{3}$

Let $u = 2x + y + 1 \qquad du = 2dx + dy$
$\quad\; v = x - y - 4 \qquad dv = dx - dy$

$du + dv = 3dx$

$dx = \frac{du + dv}{3} \qquad dy = \frac{du - 2dv}{3}$

$$u \cdot \frac{du + dv}{3} + v \cdot \frac{du - 2dv}{3} = 0$$

$$u(du + dv) + v(du - 2dv) = 0$$

$$(u + v)du + (u - 2v)dv = 0 \quad \leftarrow \text{homogeneous}$$

This only works when the two lines intersect.

Method 3: If the two lines don't intersect, we only need one substitution.

$$(x + 2y + 5)dx + (2x + 4y - 3)dy = 0$$

Suppose we tried Method 2.

Let $u = x + 2y + 5 \qquad du = dx + 2dy$
$\quad\; v = 2x + 4y - 3 \qquad dv = 2dx + 4dy$

not linearly independent so we can't solve for dx & dy.

But what if we only keep one of the new variables:

Let $u = x + 2y + 5 \qquad du = dx + 2dy$

$\Rightarrow x = u - 2y - 5 \qquad dx = du - 2dy$

$$u(du - 2dy) + (2u - 13)dy = 0 \quad \leftarrow \text{separable}$$

$$u\,du - 13\,dy = 0$$

Exact differentials

From multivariable calculus (the coreq B41), we can define the total differential of a function $z = f(x, y)$ by

$$dz = \frac{\partial f(x,y)}{\partial x}dx + \frac{\partial f(x,y)}{\partial y}dy.$$

Ex. $z = x^2 + y^2$

$$dz = 2x\,dx + 2y\,dy$$

Def. A differential expression

$$P(x,y)dx + Q(x,y)dy$$

is called an **exact differential** if it is the total differential of some function $f(x,y)$.

of some function $f(x,y)$.

i.e. if $P(x,y) = \frac{\partial}{\partial x} f(x,y)$ and $Q(x,y) = \frac{\partial}{\partial y} f(x,y)$.

When an exact differential appears in an ODE, and if we know the function it is the total differential of, then we can easily integrate the **exact differential equation** to get a 1-parameter family of solutions

$$f(x,y) = c.$$

Ex. $(6xy + 5y)dx + (3x^2 + 5x + 3y^2)dy = 0$

The LHS is an exact differential, the total differential of $f(x,y) = 3x^2y + 5xy + y^3$.

Thus $3x^2y + 5xy + y^3 = C$ is a solution to the ODE.

How do we recognize exact differentials?

Theorem 9.3 (Tenenbaum): $P(x,y)\,dx + Q(x,y)\,dy = 0$ is exact if and only if

$$\frac{\partial}{\partial y} P(x,y) = \frac{\partial}{\partial x} Q(x,y),$$

where $\frac{\partial P}{\partial x}, \frac{\partial P}{\partial y}, \frac{\partial Q}{\partial x}, \frac{\partial Q}{\partial y}$ all exist and are continuous in a simply connected region $R \subseteq \mathbb{R}^2$.

Proof. See Lesson 9B, Tenenbaum.

Ex. $2x\,dx + 3y\,dy = 0$

$$(2x + \sin y)dx + (x\cos y + y^2 - 5y)dy = 0$$

$$\left(\frac{2xy+1}{y}\right)dx + \left(\frac{y-x}{y^2}\right)dy = 0$$

Recognizing an exact differential is easy. But can we solve for $f(x,y)$?

Yes. (Tenenbaum 9.43) Given $P(x,y)dx + Q(x,y)dy = 0$ an exact differential,

$$f(x,y) = \int_{x_0}^{x} P(x,y)\,dx + \int_{y_0}^{y} Q(x_0,y)\,dy, \quad \text{where}$$

the line segments (x_0,y_0) — (x,y_0) and (x_0,y_0) — (x_0,y) lie entirely in R.

Ex. $$\left(\frac{2xy+1}{y}\right)dx + \left(\frac{y-x}{y^2}\right)dy = 0$$

$$\left.\begin{aligned}\frac{\partial}{\partial y}\left(\frac{2xy+1}{y}\right) &= \frac{\partial}{\partial y}\left(2x+\frac{1}{y}\right) = -\frac{1}{y^2}\\ \frac{\partial}{\partial x}\left(\frac{y-x}{y^2}\right) &= \frac{\partial}{\partial x}\left(\frac{1}{y}-\frac{x}{y^2}\right) = -\frac{1}{y^2}\end{aligned}\right\}\text{exact}$$

So long as $y>0$, all the partials exist & are continuous.

We choose $x_0=0$, $y_0=1$.

$$f(x,y) = \int_0^x \frac{2\bar{x}y+1}{y}\,d\bar{x} + \int_1^y \frac{\bar{y}-0}{\bar{y}^2}\,dy$$

$$= \int_0^x 2\bar{x}\,d\bar{x} + \int_0^x \frac{1}{y}\,d\bar{x} + \int_1^y \frac{1}{\bar{y}}\,d\bar{y}$$

$$= x^2 + \frac{x}{y} + \ln|y|$$

Thus $f(x,y) = x^2 + \frac{x}{y} + \ln|y| = c$ solves the ODE.

Also, can sometimes look up common integrable combinations in Tenenbaum, Table, Lesson 10A.

Integrating Factors

Sometimes, given an inexact $P(x,y)\,dx + Q(x,y)\,dy = 0$, we can convert it to an exact one by multiplying.

Define A multiplying factor which will convert an inexact ODE $P(x,y)dx + Q(x,y)dy = 0$ into an exact one $hP(x,y)dx + hQ(x,y)dy = 0$ is called an **Integrating Factor**

Ex. $(y^2+y)dx - x\,dy = 0$

Not exact because $\frac{\partial}{\partial y}(y^2+y) = 2y+1$, $\frac{\partial}{\partial x}(-x) = -1$.

However, if we multiply by y^{-2}, ← integrating factor

$$y^{-2}(y^2+y)\,dx - y^{-2}x\,dy = 0$$

$$\Rightarrow (1+y^{-1})dx - xy^{-2}dy = 0$$

$$\frac{\partial}{\partial y}(1+y^{-1}) = -\frac{1}{y^2}, \quad \frac{\partial}{\partial x}(-xy^{-2}) = -\frac{1}{y^2}. \quad \leftarrow \text{exact}$$

Integrating factors are generally quite hard to find, so we will not spend that much time on them, but you should be aware that a few common types of ODEs have known IFs.

Ex. (Lesson 11B) Given $\frac{dy}{dx} + P(x)y = Q(x)$, a linear differential eqn of first order,

a known IF is $e^{\int P(x)dx}$.